# Wakefield excitation in multimode structures by a train of electron bunches

J. G. Power, Wei Gai, and Paul Schoessow
*High Energy Physics Division, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439*
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We discuss wakefield excitation and propagation in dielectric structures, particularly concentrating on the case of multiple drive beam excitation in multimoded structures. We emphasize calculations of the energy loss of the drive beam train, the amplitude of the wakefield, and the relationship between power flow and stored energy in the dielectric wakefield device. We show that for a collinear multimode structure the amplitude of the wakefield generated by a bunch train is less than or equal to the wakefield generated by a single bunch of the same total charge. Furthermore, the transformer ratio $\mathcal{R}$ is shown to be always less than 2, even in the multiple drive beam case. [S1063-651X(99)03511-4]

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## I. INTRODUCTION

In general, the wakefield theorem [1] restricts the maximum accelerating field behind the drive bunch in a wakefield accelerator to less than twice the maximum retarding field inside the drive bunch, thus limiting the efficiency that can be obtained. One of the key concepts central to the physics of wakefield acceleration is that of the transformer ratio. The conventional definition is [2]

$$\mathcal{R}\equiv\frac{(\text{maximum energy gain behind the drive bunch})}{(\text{maximum energy loss inside the drive bunch})}.$$

For the case of a collinear drive and witness beam geometry device, $\mathcal{R}$ is always less than 2 except in a few special cases. An alternative definition [3],

$$\mathcal{R}\equiv\frac{(\text{maximum energy gain behind the drive bunch})}{(\text{average energy loss of drive bunch})},$$

can in fact exceed 2. In this paper we will use the conventional definition for the transformer ratio given by the first expression.

There have been a number of attempts to design wakefield schemes that provide transformer ratios $\mathcal{R}>2$ through the use of noncollinear drive beam/accelerated beam geometries [4], asymmetric drive beam axial distributions [2], nonlinear beam dynamics [5], and plasma dynamics [6]. There has recently been a proposal [3] to combine the use of a multimoded dielectric structure with a train of drive bunches to achieve $\mathcal{R}>2$. In this work the authors identify coherent spontaneous emission of Cherenkov radiation as the new mechanism through which the transformer ratio enhancement proceeds.

We show that under the assumptions of their analysis, there is no enhancement in the transformer ratio and that the accelerating gradient obtained is exactly what would be expected from linear superposition of the wakefields of the individual bunches.

## II. WAKEFIELD EXCITATION BY AN ARBITRARY AXIAL CHARGE DISTRIBUTION

In general, the wakefields in a structure with cylindrical symmetry can be found by first solving the problem for a point source. For a particle of charge $q$ located at position $z_0$ and moving with axial velocity $v$, the charge density $\rho$ is given by

$$\rho(r,z_0,t)=q\,\frac{\delta(r)}{r}\,\delta(z_0-vt). \tag{1}$$

Defining $z=z_0-vt$, an analytic solution can be found for the Green's function for the axial electric field of the form [7]

$$G_z(r,z)=\sum_{n=0}^{\infty}G_n(r)\cos(k_n z). \tag{2}$$

The $G_n(r)$ are the coefficients of the Fourier expansion of the wakefield that carry the radial dependence of the wakefield. From this point on we will only need to consider fields at $r=0$, so the explicit dependence on $r$ will be dropped.

For an arbitrary axial charge form factor $f(z)$, the longitudinal wakefield $E_z$ can be expressed as a convolution of the Green's function over the axial charge distribution

$$E_z(z)=\int_{-\infty}^{z}f(z')\sum_{n=0}^{\infty}G_n\cos[k_n(z-z')]dz', \tag{3}$$

where the normalization is such that $\int_{-\infty}^{\infty}f(z)dz=N$ is the number of electrons in the bunch, and $eN=Q$ is the total charge of the bunch.

For a train of $M$ bunches evenly spaced at multiples of $\lambda$, with $f_m(z)$ as the distribution of the $m$th bunch, the distribution can be written as

$$f(z)=\sum_{m=0}^{M-1}f_m(z-m\lambda), \tag{4}$$

and the corresponding wakefield as

 

$$E_z(z)=\int_{-\infty}^{z}\left[\sum_{m=0}^{M-1} f_m(z'-m\lambda)\right]\sum_{n=0}^{\infty} G_n\cos[k_n(z-z')]dz'$$
$$\equiv\sum_{m=0}^{M-1} E_{zm}(z), \tag{5}$$

where $E_{zm}$ is the longitudinal wakefield due to the $m$th bunch (located at $z=m\lambda$) acting alone.

This expression for the wakefield of an extended charge distribution can be evaluated most conveniently using numerical techniques. For dielectric structures an analytic solution can also be obtained.

## III. ENERGY LOSS OF A BUNCH TRAIN

Consider the $m$th bunch in the train. Assuming $v\simeq c$ so that by causality the bunch is affected only by its self-field and the fields of the bunches leading it, the energy loss per unit length traversed by the $m$th bunch can be expressed as

$$U_m/e=\int_{-\infty}^{\infty}dz\, f_m(z-m\lambda)\left\{\int_{-\infty}^{z}dz'\left[\left(\sum_{m'=0}^{m} f_{m'}(z'-m'\lambda)\right)G_z(z-z')\right]\right\}, \tag{6}$$

$$=\int_{-\infty}^{\infty}dz\, f_m(z-m\lambda)\left\{\int_{-\infty}^{z}dz'\{[f_m(z'-m\lambda)]G_z(z-z')\}\right\}+\int_{-\infty}^{\infty}dz\, f_m(z-m\lambda)$$
$$\times\left\{\int_{-\infty}^{z}dz'\left[\left(\sum_{m'=0}^{m-1} f_{m'}(z'-m'\lambda)\right)G_z(z-z')\right]\right\}, \tag{7}$$

$$=\int_{-\infty}^{\infty}dz\, f_m(z-m\lambda)\left\{\int_{-\infty}^{z}dz'\{[f_m(z'-m\lambda)]G_z(z-z')\}\right\}+\int_{-\infty}^{\infty}dz\, f_m(z-m\lambda)\left\{\sum_{m'=0}^{m-1} E_{zm'}(z)\right\}. \tag{8}$$

The first term is the energy loss due to the self-wakefield, while the second term is the energy loss due to the wakefields of the $(m-1)$ leading bunches. Also, note that

$$\left|e\int_{-\infty}^{\infty}dz\, f_m(z)\cos(k_n z)\right|<|Q|. \tag{9}$$

It is in the derivation of Eq. (8) that the authors of [3] made their error; they obtained $\Sigma_{m'=0}^{m-1}E_{zm'}(m\lambda)$ for the last term in Eq. (8) [their equation (17)] rather than the correct expression

$$\int_{-\infty}^{\infty}dz\, f_m(z-m\lambda)\left\{\sum_{m'=0}^{m-1} E_{zm'}(z)\right\}.$$

## IV. ENERGY STORED IN THE STRUCTURE

The stored energy per unit length in the structure is given by ([7])

$$U=\frac{1}{2L}\int_{z}^{L+z}dz\int\int dx\,dy[\epsilon(x,y,z)E^2+\mu H^2] \tag{10}$$

$$=\frac{E_0^2}{2L}\int_{z}^{L+z}dz\int\int dx\,dy F(x,y,z), \tag{11}$$

where $E_0$ is the peak axial electric field and $F(x,y,z)$ depends only on the geometry of the structure. It is simpler, however, to consider the energy lost by the bunch train based on consideration of the wakefield itself. Since the axial electric field behind the $m$th bunch is given by

$$E_z=\int_{-\infty}^{z}dz'\left\{\sum_{m'=0}^{m} f_{m'}(z'-m'\lambda)G_z(z-z')\right\}, \tag{12}$$

it follows that

$$U=R\left(\sum_{m=0}^{M-1}\sum_{n=0}^{\infty}\eta_{nm}E_n\right)^2=R\left(\sum_{m=0}^{M-1} W_m\right)^2, \tag{13}$$

where $\eta_{nm}$ is a form factor matrix depending on the detailed bunch distribution, $W_m$ is the peak wakefield left behind by the $m$th bunch, and R is a geometric factor that can be determined either by direct integration or by the method described in the following section.

## V. CONSERVATION OF ENERGY

By conservation of energy, the stored energy per unit length must equal the energy loss by all preceding bunches minus the energy flow out of the volume due to the propagating wakefield itself, i.e.,

$$U=R\left(\sum_{m=0}^{M-1} W_m\right)^2=\sum_{m=0}^{M-1} U_m+\sum_{m=0}^{M-1} P_m/c. \tag{14}$$

Here

$$P_m=\frac{c}{8\pi}\int\int E_r^m H_\phi^m dS \tag{15}$$

is the power flow (integral of the Poynting flux over the surface of the volume under consideration). The power flow is related to the stored energy per unit length ([7]) by

$$U=\sum_{m=0}^{M-1} P_m/v_g, \tag{16}$$

where $v_g$ is the group velocity of the wave. Equation (14) can be rearranged to give

$$\sum_{m=0}^{M-1} U_m=(1-v_g/c)U, \tag{17}$$

where $U_m$ is given by Eq. (8).

Since the energy loss of the first bunch in the train can be calculated using Eq. (8), the above equation can be used to obtain the wakefields of subsequent bunches, as in Ref. [3]. However, the $U_m$ depends on both the self-wakefield and the wakefield generated by the preceeding bunches and, most importantly, on the detailed axial distribution of the $m$th bunch. The effect of the power flow term is small but not negligible, as will be demonstrated by the numerical calculations below.

## VI. THE WAKEFIELD OF A TRAIN OF GAUSSIAN BUNCHES

We specialize to the case where the charge distribution is a set of $M$ identical equally spaced Gaussian bunches. The axial charge distribution is then

$$f(z)=\frac{N}{\sqrt{2\pi}\sigma_z}\sum_{m=0}^{M-1} e^{-(z-m\lambda)^2/2\sigma_z^2}. \tag{18}$$

The energy loss (per unit length) of the first bunch is then

$$U_0/e=\int_{-\infty}^{\infty} dz\, f_0(z)\int_{-\infty}^{z} dz'\, f_0(z')\times\left\{\sum_{n=0}^{\infty} G_n\cos[k_n(z-z')]\right\} \tag{19}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2}, \tag{20}$$

where we have used the relation

$$\int_{-\infty}^{\infty} dz\, f(z)\int_{-\infty}^{z} dz' f(z')\cos k_n(z-z')=\frac{1}{2}e^{-(k_n\sigma_z)^2} \tag{21}$$

and Eq. (8).

The wakefield left behind this bunch for $z\gg\sigma_z$ can be easily obtained as

$$E_z(z)=\int_{-\infty}^{z} dz' f(z')\sum_{n=0}^{\infty} G_n\cos[k_n(z-z')] \tag{22}$$

$$=\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2/2}\cos(k_n z). \tag{23}$$

For a perfect harmonic structure, defined as one for which the wakefield modes are equispaced, i.e., occur at wave numbers $k_n=(2\pi n)/\lambda$, at $z=\lambda$ ($\cos k_n z=1$) the maximum wakefield on axis is

$$E_1=\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2/2}. \tag{24}$$

Using the energy balance equation (14) we can solve for $R$:

$$R=\frac{\frac{1}{2}\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2}}{\left(\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2/2}\right)^2}. \tag{25}$$

Similarly, we can calculate the wake amplitude after the second bunch again using the energy balance equation (14)

$$R(E_1+E_2)^2=U_0+U_1, \tag{26}$$

where

$$U_1/e=\int_{-\infty}^{\infty} dz\, f_1(z)\int_{-\infty}^{z} dz' f_1(z')\times\left\{\sum_{n=0}^{\infty} G_n\cos[k_n(z-z')]\right\}+\int_{-\infty}^{\infty} dz\, f_0(z)\left\{\sum_{n=0}^{\infty} G_n\cos[k_n(z-z')]\right\} \tag{27}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2}+\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2}. \tag{28}$$

Using Eqs. (25) and (26), we get

$$E_1+E_2=\left\{\frac{\frac{1}{2}\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2}+\frac{3}{2}\sum_{n=0}^{\infty} G_n e^{-(k_n\sigma_z)^2}}{R}\right\}^{1/2}, \tag{29}$$

$$=2E_1. \tag{30}$$

Therefore, $E_2=E_1$. Following the same reasoning, one can also show that $E_1=E_2=\cdots=E_M$. In other words, the contribution to the wakefield amplitude from each beam is independent of the beams preceding it. This is what one would expect from linear superposition of the fields; in particular, there are no ‘‘stimulated emission’’ effects present.

## VII. THE TRANSFORMER RATIO AND WAKEFIELD DEVICE PHYSICS

In this section we consider the details of the circumstances under which $\mathcal{R}>2$ can be obtained, and whether these effects are relevant for wakefield devices driven by relativistic beams. We will specialize the discussion to collinear devices, where the drive and accelerated beams follow

TABLE I. Wakefield amplitudes for the harmonic planar structure described in the text. $E_{w,i}$ is the peak individual contribution of each bunch to the net wakefield. Our analysis shows no stimulated emission so this value is the same for each bunch. $\Sigma_i E_{w,i}$ (perfect harmonic) is the sum of the contributions of the $i$th bunch and preceeding bunches to the wakefield in the approximation of equal mode spacing, and is equal to $i \times E_{w,i}$ $(k_n z = 1)$. The column labeled $\Sigma_i E_{w,i}$ (simulation) shows the calculated peak wakefield taking into account the actual mode spectrum of the structure. All fields are in units of MV/m.

| Bunch number | $E_{w,i}$ | $\Sigma_i E_{w,i}$ (perfect harmonic) | $\Sigma_i E_{w,i}$ (simulation) |
|---|---|---|---|
| 1 | 5.251 | 5.251 | 5.251 |
| 2 | 5.251 | 10.502 | 10.441 |
| 3 | 5.251 | 15.753 | 15.533 |
| 4 | 5.251 | 21.004 | 20.512 |
| 5 | 5.251 | 26.255 | 25.378 |
| 6 | 5.251 | 31.506 | 30.133 |
| 7 | 5.251 | 36.757 | 34.784 |
| 8 | 5.251 | 42.008 | 39.345 |
| 9 | 5.251 | 47.259 | 43.813 |
| 10 | 5.251 | 52.510 | 48.195 |

identical trajectories through the wakefield structure. Noncollinear devices, where the two beams follow different paths and experience different impedances, can exhibit $\mathcal{R}$ much larger than 2 [4,8,9].

Other possibilities such as nonlinear media effects [6,10] and the use of asymmetric drive beams with appropriately tailored longitudinal current distributions [2] are also not relevant here, since the authors of Ref. [3] do not make use of these effects in their analysis. This leaves particle redistribution or *mixing* [5] as a possible source of transformer ratio enhancement.

The central idea is to use a relatively long drive beam ($\sigma_z \sim \frac{1}{2} \lambda_{wake}$); the tail of the bunch would gain energy while the head loses energy, so that the velocity difference would cause the tail to overtake the head. Because the velocity difference would be small for a relativistic electron beam, the mixing process would require a very long propagation distance to develop, unless heavier particles such as protons were used. Ideally, through the mixing process the average energy gain of the accelerated beam can be more than a factor of 2 over the energy of the drive beam.

To put this into perspective, consider a 30 MeV electron beam with 50% momentum spread due to its self-wake. Because the tail portion has a higher energy, it will overtake the head but it would require more than 10 m to do this. However, in 10 m a 30 MeV beam would dissolve even assuming a modest gradient of 5 MeV/m. We conclude that mixing is not a viable mechanism for enhancing the transformer ratio in electron-beam-driven wakefield devices.

## VIII. NUMERICAL SIMULATIONS OF MULTIMODE, MULTIBEAM WAKEFIELD DEVICES

In this section we show our numerical results for the example given in Ref. [3]. The device geometry is planar; two dielectric slabs (dielectric constant $\epsilon = 10$ and thickness 0.847 cm) are separated by a vacuum gap of half-height $a = 0.3$ cm. The exterior of the device is assumed to be perfectly conductive. The beam is a train of 10 Gaussian bunches, with a charge density of 2 nC/mm/bunch transverse to the direction of motion.

The Green's function for this structure is given in Eq. (7) of [3]. Using this Green's function, we reproduce nearly the

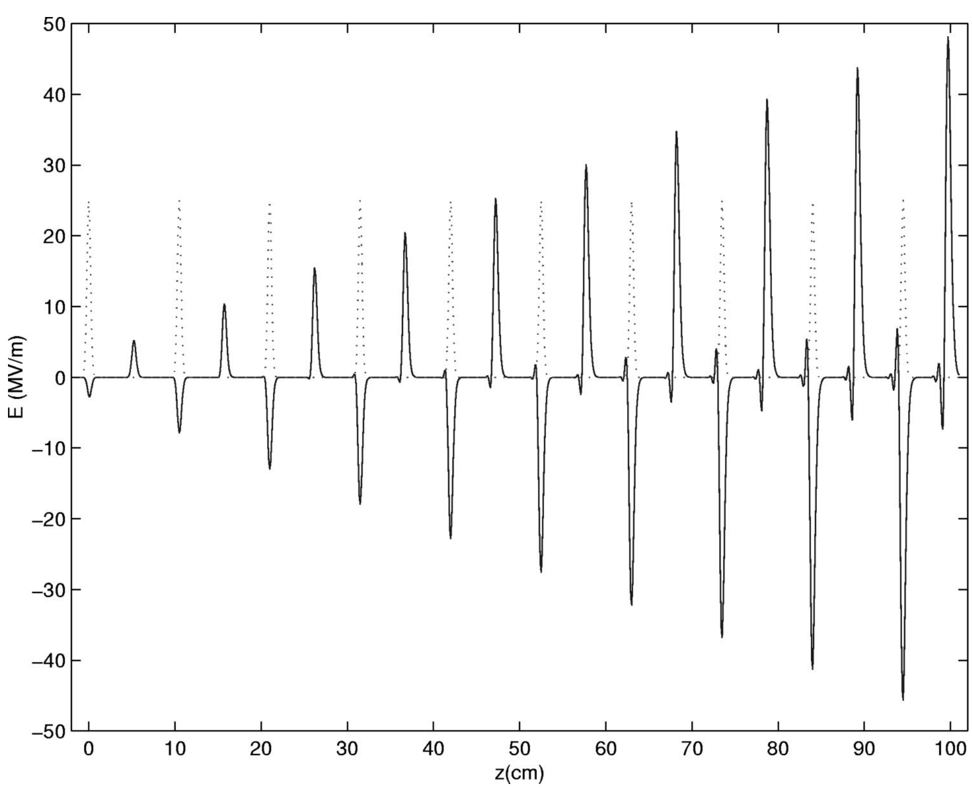


FIG. 1. Analytic wakefield of a bunch train in the dielectric loaded slab structure described in Ref. [3]. Dotted lines show the beam profile.

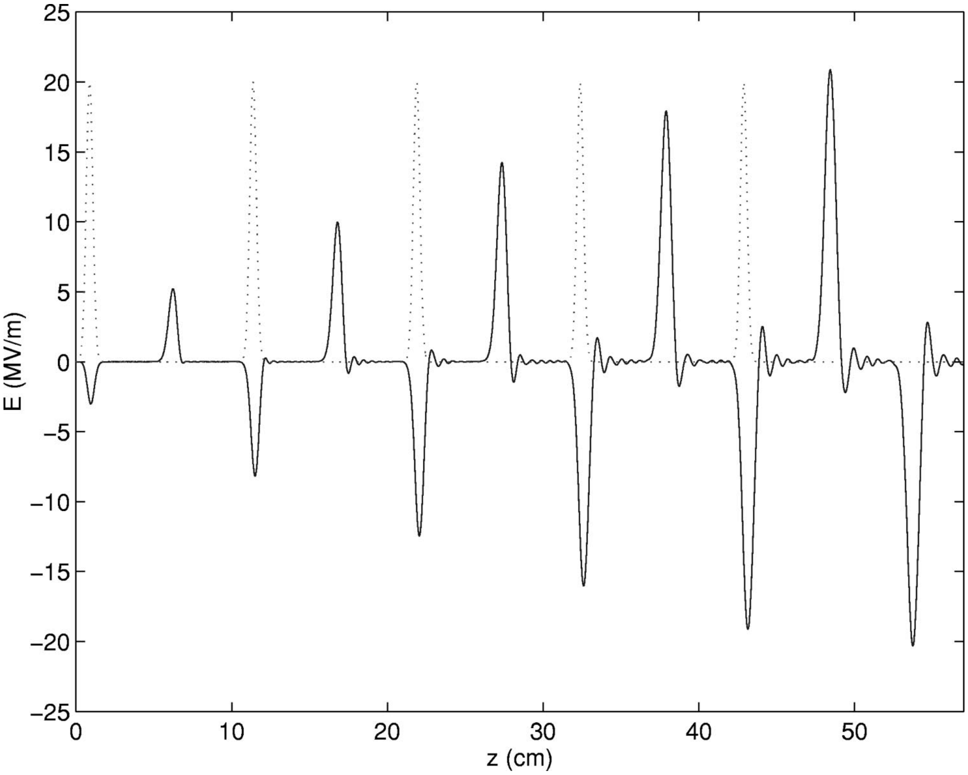


FIG. 2. Wakefield of a bunch train in the dielectric loaded slab structure described in [3] calculated by direct numerical integration of Maxwell's equations. The sizes of successive accelerating peaks scale linearly with bunch number; no stimulated emission effects are observed. Small discrepancies with the analytic calculation are due to finite mesh-size effects.

same wakefield amplitude after the leading bunch (Table I). However, using the expressions derived for multiple drive bunches in Sec. VI, the subsequent wakefield amplitudes diverge from the results of Ref. [3].

The wakefield produced by a train of Gaussian pulses is shown in Fig. 1, as computed from the mode summation formula [Eq. (5)] derived in this paper. The wake amplitude at a given position is always less than or equal to the sum of the maximum wake amplitudes of the bunches leading it, as expected from linear superposition. Figure 2 shows a finite difference time domain calculation of the same problem, where the Maxwell equations are discretized on a mesh and the fields evolved in the time domain. Again no transformer ratio enhancements beyond those expected from simple addition of the fields from the individual bunches can be observed.

We have also evaluated the wakefield in a multimode cylindrical dielectric structure. As pointed out in Ref. [3], a

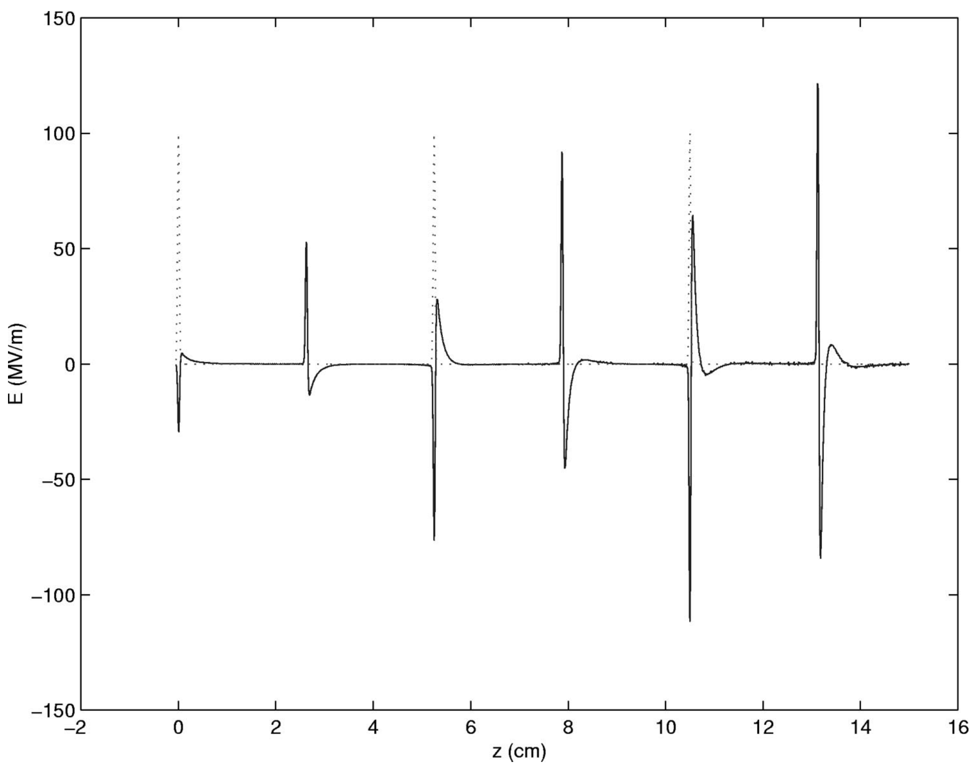


FIG. 3. Wakefield of a bunch train in an approximately harmonic dielectric loaded cylindrical structure. The peak accelerating field amplitude after the first bunch is 54.7 MV/m, and after the third bunch it is 126 MV/m $<3\times54.7$ MV/m. The less-than-linear scaling of the peak field is due to the imperfect harmonicity of the structure. In this case the transformer ratio ($\mathcal{R}\simeq55$ MV/m)/(35 MV/m $\simeq1.6$).

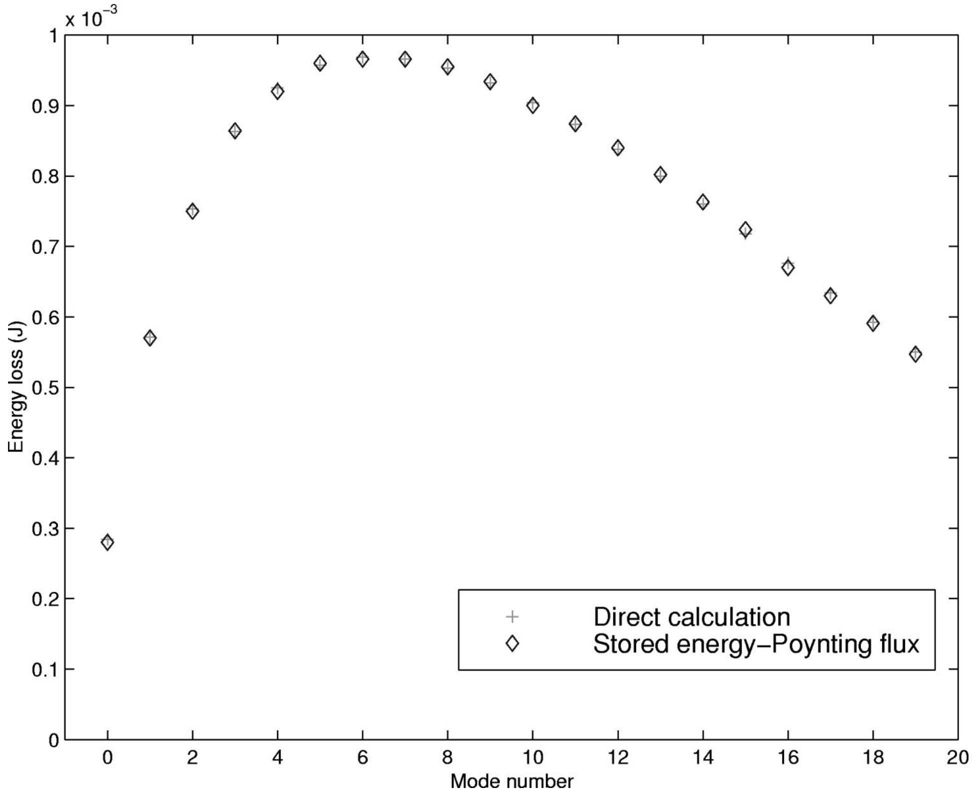


FIG. 4. Comparison of energy loss by the drive beam using direct calculation [Eq. (8)] and (stored energy)−(energy flow) [Eq. (14)].

cylindrical cavity will deviate more from perfect harmonicity than a planar device. The wakefield for the cylindrical structure described in [3] is shown in Fig. 3. The structure dimensions are inner radius 0.375 mm, outer radius 4.88 mm, and $\epsilon = 9.43$. The rms beam bunch length is 0.18 mm, and the charge is 1 nC/bunch. Again no transformer ratio enhancement is observed.

For completeness, we have also calculated the energy loss of a single drive bunch into each mode for the first 20 modes of this structure by applying Eq. (8). For comparison, we also calculated the energy stored per unit length minus the energy transported away by the Poynting flux in each mode [Eq. (14)] for a given peak axial electric field. As shown in Fig. 4, the energy loss obtained by direct calculation agrees within numerical errors with the calculation based on conservation of energy. Thus by knowing the energy stored per unit length in each mode, one can calculate the peak electric field of each mode accurately using the methods in Sec. V.

## IX. SUMMARY

We have studied the theory of wakefield excitation in dielectric structures driven by multiple electron bunches. Using arguments based on linear superposition and energy conservation, we have found that, applying the theory correctly, both methods provide independent and identical results. We have also examined the transformer ratio problem and found that it is not possible to enhance the transformer ratio by simply using multiple drive beams.

This type of structure may turn out to be useful for particle acceleration, based on the technology of generating high quality drive bunch trains from photoinjectors, less likelihood of dielectric breakdown, and for planar structures, suppression of the single bunch beam breakup instability [11]. Nevertheless, we have shown that the wakefield of a bunch train is what would be expected by linear superposition of the wakes of the individual bunches, and that no stimulated emission effects are present to enhance the transformer ratio.

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